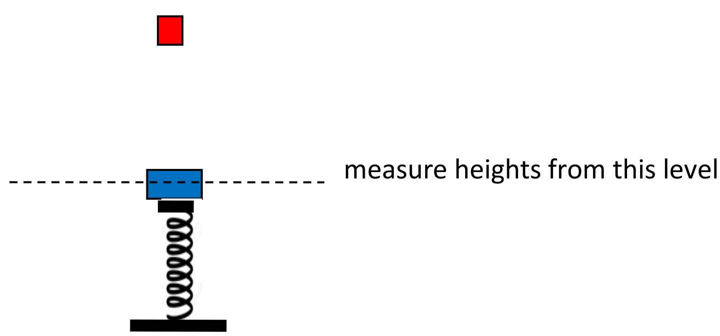
Teacher notes
Topic A

A neat problem in energy conservation

A blue body of mass M rests attached to a vertical spring that is firmly fixed to the floor. A red body of mass m falls from rest from a certain height above the blue body. The gravitational potential energy of the red body at the release height is E. The bodies collide elastically and the red body bounces to ¼ of its initial height. The zero of gravitational potential energy is taken to be the doted horizontal line.

What is the energy of the blue body right after the collision?



We will solve this problem in two ways. The first uses the formulas for the speeds of two bodies after the collide elastically. So, it uses material off our syllabus. It also involves a lot of work with algebra. Then we will use a different method which is far superior and fully within our syllabus.

Method 1 (Brute force and detailed):

First, we find the speed of the falling body at impact: we have that $\frac{1}{2}mu^2 = E$ and so $u = \sqrt{\frac{2E}{m}}$.

The collision is elastic so the speeds of the two bodies after the collision will be:

Red: $u' = \frac{M-m}{M+m}\sqrt{\frac{2E}{m}}$ upwards

Blue: $v' = \frac{2m}{M+m}\sqrt{\frac{2E}{m}}$ downwards

The red body reaches a quarter of its initial height and the potential energy there is $\frac{E}{4}$ and so

$u' = \sqrt{\frac{2E}{4m}} = \sqrt{\frac{E}{2m}}$. This means that

$\frac{M-m}{M+m}\sqrt{\frac{2E}{m}}=\sqrt{\frac{E}{2m}}$ i.e. $\frac{M-m}{M+m}=\frac{1}{2}\Rightarrow M+m=2(M-m)$ and so $M=3m$.

The energy of the blue body is then

$$\frac{1}{2}M(\frac{2m}{M+m})^2\frac{2E}{m}=\frac{1}{2}3m\frac{4m^2}{16m^2}\frac{2E}{m}$$
$$=\frac{3E}{4}$$

BUT, there is an easier and much better way!

Method 2:

The red body has kinetic energy *E* when it hits the blue body. Since it bounces to a fourth of the initial height its kinetic energy is $\frac{E}{4}$ after the collision. The blue body has zero energy before the collision, and we want to find its energy E' after the collision. By energy conservation:

$$E+0=\frac{E}{4}+E' \;\Rightarrow\; E'=\frac{3E}{4} \text{ (!)}$$

Comments

1. We are evaluating everything at the dotted line so no gravitational potential energy needs to be calculated there.
2. The collision is assumed to be instantaneous so the spring has no time to compress, hence we do not worry about elastic potential energy.
3. In method 1, the formulas for speeds are derived using conservation of momentum and kinetic energy. Why is momentum conserved? Because the collision takes place instantaneously, spring and gravity do not have time to provide any impulse.

Moral of the story: Math is great but simple physical reasoning is better.

Extension

What is the motion of the blue body after the collision? (Assume that the red body has disappeared.)

The blue body will execute simple harmonic oscillations about the equilibrium position with total energy equal to $\frac{3E}{4}$. If the spring constant is *k* then the amplitude *A* will be found from $\frac{1}{2}kA^2=\frac{3E}{4}\Rightarrow A=\sqrt{\frac{3E}{2k}}$

and the displacement will be given by $x = \sqrt{\frac{3E}{2k}}\sin(\omega t)$ (the collision takes place at t = 0 and the down direction is taken as positive) with $\omega^2 = \frac{k}{M}$. The maximum velocity is $v_{\max} = \omega\sqrt{\frac{3E}{2k}} = \sqrt{\frac{k}{M}}\sqrt{\frac{3E}{2k}} = \sqrt{\frac{3E}{2M}}$.

The maximum kinetic energy is then $E_{\max} = \frac{1}{2}M \times \frac{3E}{2M} = \frac{3E}{4}$ as we know it should be.

Challenge

Suppose the red body does not disappear. Then, after reaching the maximum height it will fall again. When does it collide with the blue body?

The position of the red ball is given by $x_{\text{red}} = -\frac{1}{2}\sqrt{\frac{E}{2m}}t + \frac{1}{2}gt^2$. That of the blue ball is $x_{\text{blue}} = \sqrt{\frac{3E}{2k}}\sin(\omega t)$.

They meet when

$$-\frac{3-1}{4}\sqrt{\frac{E}{2m}}t + \frac{1}{2}gt^2 = \sqrt{\frac{3E}{2k}}\sin(\omega t)$$

This can only be solved numerically!

So, if m = 1 kg, E = 2 J, k = 1200 N m^{-1}, M = 3 kg and $\omega = \sqrt{\frac{k}{M}} = \sqrt{\frac{1200}{3}} = 20\text{ s}^{-1}$ we have

$$-\frac{1}{2}t + 5t^2 = \frac{1}{20}\sin(20t)$$

The solution is $t = 0.133705\text{ s}$. The position is $x_{\text{red}} = -\frac{1}{2} \times 0.133705 + 5 \times 0.133705^2 = 2.25\text{ cm}$. (The amplitude is $A = \sqrt{\frac{3E}{2k}} = \frac{1}{20} = 5.0\text{ cm}$.)